

Arithmetic progressions in Salem-type subsets of the integers

Paul Potgieter

Department of Decision Sciences, University of South Africa

P.O. Box 392, Pretoria 0003, South Africa

potgip@unisa.ac.za

Abstract

Given a subset of the integers of zero density, we define the weaker notion of the fractional density of such a set. It is shown how this notion corresponds to that of the Hausdorff dimension of a compact subset of the reals. We then show that a version of a theorem of Laba and Pramanik on 3-term arithmetic progressions in subsets of the unit interval also holds for subsets of the integers with fractional density which also satisfy certain Fourier decay conditions.

Mathematics Subject Classification: 42B05, 11B25, 28A78, 26E35

1 Introduction

The existence of 3-term arithmetic progressions in certain sets of fractional Hausdorff dimension was recently established by Laba and Pramanik [3]. They introduce Salem-type sets in $[0, 1]$, that is, sets which have a positive Hausdorff dimension and a sufficiently rapid decay for the Fourier transform of some measure on the set. The origins of this theorem can be traced back to Roth's original theorem establishing 3-term arithmetic progressions in dense subsets of the integers [7]. For cases where the density of the subset is not positive, the conclusion of Roth's theorem may still hold, providing the sets are "random enough", such as is the case with the primes [2]. We will appropriate the term "Salem-type" to indicate a subset of the integers which satisfy a weak density condition as well a certain decay condition on the Fourier coefficients of its characteristic function, as specified in Theorem 4.1.

The goal of this paper is to establish a result corresponding to that of Laba and Pramanik on the integers. The first step is to formulate a version of Hausdorff dimension for sets which have zero density in the conventional sense. This allows us to relax the uniformity conditions on sets of density zero, such as discussed in [8].

In the second section we discuss the results that inspired this paper. This involves a correspondence between certain subsets of \mathbb{N} and subsets of $[0, 1]$. These were originally explored by Leth [4]. In [6], a nonstandard counting formulation of Hausdorff dimension is established. Since this formulation, when considered in the context of subsets of the natural numbers instead of compact subsets of \mathbb{R} , resembles the usual definition of density very closely, it seemed likely that a weaker idea of density would prove useful in studying arithmetic progressions, especially in the light of [3]. Indeed, when subsets of \mathbb{N} are mapped to subsets of $[0, 1]$ via a mapping similar to that in [4], this "fractional density" is preserved as Hausdorff dimension. Similarly, when a subset of $[0, 1]$ is mapped into \mathbb{N} , Hausdorff dimension is preserved in the guise of fractional density.

The third section discusses a uniformity condition (see for instance [8], p161) necessary for a set of fractional density to contain a 3-term arithmetic progression. In the fourth section, a version of

Laba and Pramanik's result is proved for subsets of \mathbb{N} . The proof involves little else but repeated use of Varnavides's theorem, as found in [8]. In the final section we construct an example of a set in the integers, analogous to that found in Section 6 of [3], which satisfies the conditions of Theorem 4.1 of this paper.

Some background in nonstandard analysis is required for the second section of this paper. A succinct but sufficient introduction to all the necessary concepts can be found in [6]. Apart from Definition 2.2, the rest of the paper can be read independently of this section. However, in order to understand the motivation behind the formulation and the direction of future investigations, it would benefit the reader to at least give it a cursory glance.

The author would like to thank Claudius du Plooy for his generous support during the conception of this paper, as well as for many enlightening conversations throughout the years.

2 Correspondence between subsets of \mathbb{N} and $[0, 1]$

We use the notation of [6] throughout. Let $A = (a_n)_{n \in \mathbb{N}}$ denote a sequence of natural numbers (which we assume to be strictly increasing). Note that in the paper [4] we are not restricted to sequences in \mathbb{N} , but it will suffice for our purposes. The essential idea behind the correspondence is to use a hyperfinite number to divide every member of the nonstandard extension *A of the sequence A (throughout this section we denote nonstandard extensions of sets similarly). The standard part of a nonstandard number or set x shall be denoted by $\text{st}(x)$. General results in [4] hold for division by any hyperfinite number $z \in {}^*\mathbb{N} \setminus \mathbb{N}$. We shall however only consider division of each element of *A by the number $\langle a_n \rangle_{\mathcal{U}}$, that is, the unique hyperfinite number determined by the sequence (a_n) under the equivalence relation of a certain (fixed) free ultrafilter \mathcal{U} (the choice of ultrafilter is immaterial to the results). We formalise this previous by defining:

Definition 2.1. *Suppose $A = (a_n)_{n \geq 1}$ is an increasing sequence of natural numbers. Then we denote by $st_z(A)$ the set*

$$\{\text{st}(a/z) : a \in {}^*A\}$$

where $z = \langle a_n \rangle_{\mathcal{U}}$.

(The definition can of course be extended from \mathbb{N} to \mathbb{Z} .) It is clear that $st_z(A) \subseteq [0, 1]$. Furthermore, it is a closed set, as shown in Proposition 2.2 of [4]. Our purpose is now to show that the Hausdorff dimension of $st_z(A)$ coincides with the “fractional density” of A . Throughout the paper we will use $[A, B]$ to denote the interval in \mathbb{Z} given by the set $\{A, A + 1, \dots, B\}$. The intervals $[A, B)$ and $(A, B]$ are defined analogously.

Definition 2.2. *We say that a set $A \subseteq \mathbb{N}$ has fractional upper density α if*

$$\limsup_{N \rightarrow \infty} \frac{|A \cap [1, N]|}{N^\beta}$$

is ∞ for any $\beta < \alpha$ and 0 for any $\beta > \alpha$.

(The lower fractional density can be similarly defined by replacing \limsup in the above with \liminf . If upper and lower fractional densities are equal, we can just speak of the fractional density.)

We can summarise this by saying that $\bar{d}_f(A) = \alpha$. We will sometimes need to consider the fractional density relative to a finite but arbitrarily large number; that is, we will say that $A \subseteq [0, N)$

has upper fractional density β relative to N if $|A|/N^\beta = c > 0$, for arbitrarily large N . In sections 3 and 4, this is mostly how the concept of upper fractional density will be utilised. Note that the limit in the above definition is the same as the limit of n/a_n^β , which is the form we will use in proving Proposition 2.2.

Of course, one has to verify that such a concept yields information that the usual definition of density does not, in the same way that Hausdorff dimension (denoted by \dim_H) yields information that Lebesgue measure does not. Firstly, it is easily verified that any subset of \mathbb{N} of positive density has fractional density 1. One can also verify that there exist sets which do not have positive density but do have positive fractional density. For example, one can create a version of the triadic Cantor set on \mathbb{N} as follows:

1. Let C_0 be the interval $(0, 3^0]$ (in \mathbb{N}). We recognise only the right-hand endpoint of the interval, leaving $C_0 = \{1\}$.
2. Let C_1 consist of the interval $[0, 3^1]$. Remove the middle third $(1, 2]$, and keep 1 and 3, the right-hand endpoints of the remaining intervals. Thus, $C_1 = \{1, 3\}$
3. Similarly with the interval $(0, 3^2]$, we remove the middle third intervals $(1, 2]$, $(3, 4]$, $(4, 5]$, $(5, 6]$ and $(7, 8]$. Thus, $C_2 = \{1, 3, 7, 9\}$, and so on.

This construction can be formalised thus:

$$C_0 = \{1\} \tag{2.1}$$

$$C_{i+1} = C_i \cup \{3^{i+1} + 1 - c : c \in C_i\}, \quad i \in \mathbb{N} \tag{2.2}$$

$$C = \bigcup_{i=0}^{\infty} C_i. \tag{2.3}$$

It is trivial to show that this set has fractional density $\log 2 / \log 3$ simply by counting elements at every stage, even though it does not have positive density.

Instead of utilising the standard definition of Hausdorff dimension, we use the following non-standard version [6]. Note that for some infinitesimal $\Delta t = 1/N$, $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, we call the set $\{0, \Delta t, 2 \Delta t, \dots, 1 - \Delta t\}$ the *hyperfinite time line* based on Δt . The function $|\cdot|$ denotes the transferred cardinality function.

Theorem 2.1. *Consider a hyperfinite time line \mathbf{T} based on the infinitesimal N^{-1} , for a given $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Suppose that an internal subset A' of the time line is such that ${}^\circ(A') = A$ and for some $\alpha > 0$*

$${}^\circ\left(\frac{|A'|}{N^\beta}\right) > 0 \text{ for } \beta < \alpha \text{ and} \tag{2.4}$$

$${}^\circ\left(\frac{|A'|}{N^\beta}\right) = 0 \text{ for } \beta > \alpha. \tag{2.5}$$

Then $\alpha = \dim A$.

One might be concerned that the nonstandard formulation of Hausdorff dimension might too closely resemble Minkowski dimension. However, as is shown in [6], this formulation implies the existence of a positive measure on a set of positive Hausdorff dimension, which is not necessarily a property of sets of positive Minkowski dimension.

A simple argument using the transfer principle now shows that fractional density of the set A is exactly the same as the Hausdorff dimension of the set $st_z(A)$.

Proposition 2.2. *Suppose that a sequence $(a_n) = A \subseteq \mathbb{N}$ has fractional density α . If $z = \langle a_n \rangle_{\mathcal{U}}$, then $st_z(A)$ has Hausdorff dimension α .*

Proof. If $\beta < \alpha$, the sequence n/a_n^β will diverge as $n \rightarrow \infty$. Hence we can assume that for all i after a certain stage, $i/a_i^\beta > 1$. If we now let a_J denote the element of the nonstandard extension of the sequence determined by the sequence itself (modulo the free ultrafilter), the property $J/a_J^\beta > 1$ will also hold by the transfer principle. Considering the set $\{a/a_J : a \in {}^*A\}$, we see that it is a subset of the hyperfinite time line based on a_J , since each element of *A is still a member of ${}^*\mathbb{N}$ (by transfer). Furthermore, Theorem 2.1 now implies that $\dim_H(st_{a_J}(A)) > \beta$. Similarly, for any $\beta > \alpha$, we obtain that $\dim_H(st_{a_J}(A)) < \beta$, concluding the proof.

The converse of the previous proposition can also be easily shown by reversing the argument, i.e. that given a subset of $[0, 1]$ of Hausdorff dimension α , we can multiply by a hyperfinite natural number (which is not unique) to obtain a set with fractional density α . A more interesting question concerns the relationship between the Fourier-dimensional properties of compact sets in \mathbb{R} and the properties of discrete Fourier coefficients of characteristic functions of analogous subsets of \mathbb{Z} . It is this relationship we are attempting to explore by interpreting the results in [3] in the context of the whole numbers.

3 Fourier conditions

The essence of the proof of Roth's theorem, as presented in e.g. [5], is to show that the Fourier transform of the characteristic function of a set of positive density either satisfies certain decay conditions, or the set has increased density in some arithmetic progression in \mathbb{Z} . Iterating this argument on the assumption that the set contains no 3-term arithmetic progressions, a density of greater than 1 is eventually obtained on some arithmetic progression, a contradiction.

If the set does not have positive density, we have to impose decay conditions on the Fourier coefficients. We first determine the uniform rate of decay necessary to guarantee such progressions when a set has fractional density $\alpha < 1$. We will say that a subset A of a finite additive group Z is γ -uniform if the Fourier coefficients of the characteristic function satisfy $|\widehat{\chi_A}(k)| \leq \gamma$ for all $k \in Z$, $k \neq 0$. If this γ is small, the set is said to be linearly uniform. In the case of a set of positive density, it is possible to find linear uniformity conditions which guarantee the existence of progressions. Our version of this will be to find some β such that if the Fourier coefficients are all smaller than cN^β for some c , we will be guaranteed a 3-term arithmetic progression.

Consider $A \subset \mathbb{Z}$ such that for some $0 < \alpha < 1$, $|A \cap [0, N]| \geq \delta N^\alpha$ for arbitrarily large N . (This implies that the upper fractional density of A is $\geq \alpha$.) We will assume, without loss, that $|A \cap [0, N]| = \delta N^\alpha$ for each N under consideration. As a first approximation to the 3-term arithmetic progressions contained $A \cap [0, N)$, we count the number of progressions modulo N , i.e. the number of $x, y, z \in A$ such that

$$x + y \equiv 2z \pmod{N}.$$

(In this we follow Lyall's exposition of Roth's theorem [5], and use similar notation.) The Fourier coefficients of a function defined on the integers modulo N (denoted by \mathbb{Z}_N) are defined as usual

by

$$\hat{f}(k) = \frac{1}{N} \sum_{n=0}^{N-1} f(x) e^{-\frac{2\pi i k n}{N}}$$

The number of triples satisfying the congruence, if χ_A denotes the characteristic function of A , is given by

$$\mathcal{N}_0 = N^2 \sum_{n=0}^{N-1} \widehat{\chi_A}(n) \widehat{\chi_A}(n) \widehat{\chi_A}(-2n)$$

However, a triple satisfying the congruence does not necessarily form a true arithmetic progression in \mathbb{Z} , since some of the terms might “wrap around” the cyclic group. If we require instead that $x, z \in M_A = A \cap [N/3, 2N/3]$, then a \mathbb{Z}_N -progression does indeed form a \mathbb{Z} -progression. In this case, we estimate the true triples \mathcal{N} by writing

$$\mathcal{N} \geq N^2 \sum_{n=0}^{N-1} \widehat{\chi_{M_A}}(n) \widehat{\chi_A}(n) \widehat{\chi_{M_A}}(-2n) = \delta N^{\alpha-1} |M_A|^2 + N^2 \sum_{n=1}^{N-1} \widehat{\chi_{M_A}}(n) \widehat{\chi_A}(n) \widehat{\chi_{M_A}}(-2n).$$

We require that $|M_A| \geq \frac{\delta}{4} N^\alpha$ and $|\widehat{\chi_A}(k)| \leq \delta^2 N^\beta / 32$ for $k \neq 0$. Using the Cauchy-Schwartz inequality, this gives

$$\begin{aligned} N^2 \left| \sum_{n=1}^{N-1} \widehat{\chi_{M_A}}(n) \widehat{\chi_A}(n) \widehat{\chi_{M_A}}(-2n) \right| &\leq N^2 \max_{k \neq 0} |\widehat{\chi_A}(k)| \left| \sum_{n=1}^{N-1} \widehat{\chi_{M_A}}(n) \widehat{\chi_{M_A}}(-2n) \right| \\ &\leq N^2 \max_{k \neq 0} |\widehat{\chi_A}(k)| \left(\sum_{n=1}^{N-1} |\widehat{\chi_{M_A}}(n)|^2 \right)^{\frac{1}{2}} \left(\sum_n |\widehat{\chi_{M_A}}(-2n)|^2 \right)^{\frac{1}{2}} \\ &\leq N^2 \max_{k \neq 0} |\widehat{\chi_A}(k)| \sum_{n=0}^{N-1} |\widehat{\chi_{M_A}}(n)|^2 \\ &= N^2 \max_{k \neq 0} |\widehat{\chi_A}(k)| \cdot \frac{1}{N} \sum_{x=0}^{N-1} \chi_{M_A}(x) \\ &\leq \frac{\delta^2}{32} N^\beta \cdot N \cdot |M_A| \\ &\leq \frac{\delta^3}{32} N^\beta N^{\alpha+1} \end{aligned}$$

If we now require that $\beta < 2\alpha - 2$, say $\beta = 2\alpha - 2 - \varepsilon$, we find that

$$\begin{aligned} \mathcal{N} &\geq \delta N^{\alpha-1} |M_A|^2 - \frac{\delta^3}{32} N^{3\alpha-1-\varepsilon} \\ &\geq \frac{\delta^3 N^{3\alpha-1}}{32} (2 - N^{-\varepsilon}) \\ &> \frac{\delta^3 N^{3\alpha-1}}{32} \end{aligned}$$

This will be large for $\alpha > 1/3$.

We still have not taken into account the number of trivial progressions $x = y = z$, of which there are $|A| = \delta N^\alpha$. If we subtract this from the estimate obtained above and require that $\alpha > 1/2$ and (for instance) $N > 32/\delta^2$, we are certain to have a non-trivial 3-progression.

Of course, we might not always be as fortunate as to have such small Fourier coefficients. In the next section we show that weaker non-uniform conditions would still suffice, provided that the decay is sufficiently structured.

4 Salem-type sets in the integers

In this section we prove the following:

Theorem 4.1. *Let $A \subseteq \mathbb{Z}$. Suppose A satisfies the following conditions:*

- (i) *A has upper fractional density α , where $\alpha > 1/2$.*
- (ii) *The Fourier coefficients of the characteristic functions χ_{A_N} of $A_N = A \cap [0, N-1]$ satisfy*

$$|\widehat{\chi_{A_N}}(k)| \leq C(|k|N)^{-\beta/2}$$

for large N , for some $2/3 < \beta \leq 1$ satisfying $\beta > 2 - 2\alpha$.

Then A contains an arithmetic progression of length 3.

As long as the interval $[0, N-1]$ is fixed, as it is throughout most of the proof, we will use simply A instead of A_N .

To prove Proposition 4.1, we use a modified version of the density arguments using Varnavides's theorem, to be found in e.g. [8]. Throughout, we use Z to denote a finite additive group of odd order N . The expectation of a function on Z is defined as

$$\mathbf{E}_Z(f) = \mathbf{E}_{x \in Z}(f) = \frac{1}{|Z|} \sum_{x \in Z} f(x)$$

The $L^p(Z)$ -norm of a function $f : Z \rightarrow \mathbb{C}$ is given by

$$\|f\|_{L^p(Z)} = \left(\frac{1}{N} \sum_{n=0}^{N-1} |f(n)|^p \right)^{\frac{1}{p}}.$$

We also define the *linear bias* of a function $f : Z \rightarrow \mathbb{C}$ by

$$\|f\|_{u^2(Z)} = \sup_{\xi \in Z} |\hat{f}(\xi)|.$$

In the proof we will repeatedly use the following definition:

Definition 4.1.

$$\Lambda_3(f, g, h) = \mathbf{E}_{x, r \in Z} f(x) g(x+r) h(x+2r)$$

Note that $|Z|^2 \Lambda_3(\chi_A, \chi_A, \chi_A)$ is an indication of the number of 3-term arithmetic progressions to be found in a set $A \subseteq Z$, although some might be counted more than once. To remove trivial progressions, one has to subtract $|A|$. It follows that if $|Z|^2 \Lambda_3(\chi_A, \chi_A, \chi_A) - |A|$ is suitably large, A will contain at least one 3-progression as a subset of the group Z .

The following can be found in [8], p.374.

Proposition 4.2. For functions f, g and h from Z to \mathbb{C} ,

$$\Lambda_3(f, g, h) = \sum_{n=0}^{N-1} \widehat{f}(n) \widehat{g}(-2n) \widehat{h}(n). \quad (4.6)$$

We also have the following property of Λ_3 [8]:

$$\Lambda_3(f, g, h) \leq \|f\|_{u^2(Z)} \|g\|_{L^2(Z)} \|h\|_{L^2(Z)} \quad (4.7)$$

Proof of Theorem 4.1. We can assume that not all of the Fourier coefficients are smaller than or equal to $\delta^2/8N^\beta$, since that would immediately imply a 3-term arithmetic progression, by the result in Section 3.

From now on, we denote χ_A by μ , for brevity and also to consolidate the analogy with [3]. Where they consider a compact subset of $[0, 1]$ of certain Hausdorff dimension together with a sufficient decay of the measure guaranteed to exist on the set, we consider a set of fractional density with sufficient decay of the discrete Fourier transform of the characteristic function.

We decompose μ into a sum $\mu_1 + \mu_2$. Using this, we estimate the expression $\Lambda(\mu, \mu, \mu)$. If this is large enough, it will guarantee the existence of a 3-term arithmetic progression.

We let F_K denote a version of the Fejér kernel on $[0, N-1]$:

$$F_K(x) = \sum_{k=0}^K \left(1 - \frac{k}{K+1}\right) e^{\frac{2\pi i k x}{N}}.$$

Define μ_1 as the convolution of μ and F_K :

$$\mu_1(x) = (F_K * \mu)(x) = \sum_{y=0}^{N-1} \sum_{n=0}^K \left(1 - \frac{n}{K+1}\right) e^{\frac{2\pi i n(x-y)}{N}} \mu(y).$$

By rewriting the convolution product, we can find the Fourier series of μ_1 :

$$\begin{aligned} \mu_1(x) &= \sum_{k=0}^K \sum_{n=0}^{N-1} \left(1 - \frac{k}{K+1}\right) e^{\frac{2\pi i k x}{N}} e^{-\frac{2\pi i k n}{N}} \chi_A(n) \\ &= \sum_{k=0}^K \left(1 - \frac{k}{K+1}\right) e^{2\pi i k x} \widehat{\chi_A}(k). \end{aligned}$$

Thus, if $n < K+1$,

$$\widehat{\mu_1}(n) = \left(1 - \frac{n}{K+1}\right) \widehat{\chi_A}(n).$$

Otherwise, $\widehat{\mu_1}(n) = 0$. Also, since $\widehat{\mu_2}(n) = \widehat{\chi_A}(n) - \widehat{\mu_1}(n)$,

$$\widehat{\mu_2}(n) = \min\left(1, \frac{n}{K+1}\right) \widehat{\chi_A}(n).$$

To calculate $\Lambda_3(\mu, \mu, \mu)$, we split the expression $\Lambda_3(\mu_1 + \mu_2, \mu_1 + \mu_2, \mu_1 + \mu_2)$ into eight terms of the form $\Lambda_3(\mu_i, \mu_j, \mu_k)$, $i, j, k \in \{1, 2\}$. The idea is then to show that the term $\Lambda_3(\mu_1, \mu_1, \mu_1)$ dominates the others, and will be large enough to guarantee an arithmetic progression.

We can now use the following inequality, which follows from (4.2):

$$|\Lambda_3(f, g, h)| \leq \sum_{0 \leq n < N} |\hat{f}(n)| |\hat{g}(-2n)| |\hat{h}(n)|. \quad (4.8)$$

We only evaluate two of the terms which contain at most two instances of μ_1 . The others can be evaluated according to the exact same principles.

Throughout the calculations, we assume that $K < N/2$, so that, for $1 \leq n \leq K$, $|-2n|^{-\frac{\beta}{2}} \leq (2n)^{-\frac{\beta}{2}}$. Furthermore, this implies that $\min\{1, 1 - (N - 2n)/(K + 1)\} = 1$ for $1 \leq n \leq K$. We will later see that the lower bound we place on K does not violate these conditions. This assumption allows us to replace $|-2k|$ by $2k$ in the sequel.

First considering the term $\Lambda_3(\mu_1, \mu_2, \mu_1)$, we know from inequality 4.8, the fact that $\widehat{\mu}_1(n) = 0$ for $n \geq K + 1$ and $\widehat{\mu}_2(0) = 0$ that

$$\begin{aligned} |\Lambda_3(\mu_1, \mu_2, \mu_1)| &\leq \sum_{0 < n \leq N} |\widehat{\mu}_1(n)|^2 |\widehat{\mu}_2(-2n)| \\ &= O \left(N^{-\frac{3\beta}{2}} \sum_{0 < n \leq K} \left(1 - \frac{n}{K+1}\right) n^{-\frac{3\beta}{2}} \right) \\ &= O \left(N^{-\frac{3\beta}{2}} \sum_{0 < n \leq K} n^{-\frac{3\beta}{2}} \right) \\ &= O \left(N^{-\frac{3\beta}{2}} \right), \end{aligned}$$

since the sum $\sum_{n=1}^{\infty} n^{-\frac{3\beta}{2}}$ is convergent.

Next, we turn to the expression $\Lambda_3(\mu_1, \mu_2, \mu_2)$. Using the same properties of the Fourier coefficients, we find once again that

$$\begin{aligned} |\Lambda_3(\mu_1, \mu_2, \mu_2)| &\leq \sum_{0 < n \leq N} |\widehat{\mu}_1(n)| |\widehat{\mu}_2(-2n)| |\widehat{\mu}_2(n)| \\ &= O \left(N^{-\frac{3\beta}{2}} \sum_{0 < n \leq K} \left(1 - \frac{n}{K+1}\right) n^{-\frac{3\beta}{2}} \right). \end{aligned}$$

The same bound clearly applies as for the previous expression. Because the Fourier coefficients of μ_1 are 0 for $n \geq K + 1$, any term involving μ_1 can be approximated this way. If the term does not involve μ_1 , we have no such cut-off, yet even without such we can still easily obtain an upper bound of $O(N^{-\frac{3\beta}{2}})$ on $|\Lambda_3(\mu_2, \mu_2, \mu_2)|$.

Hence, all terms in the expansion of $\Lambda_3(\mu_1 + \mu_2, \mu_1 + \mu_2, \mu_1 + \mu_2)$ that involve μ_2 become at most $O(N^{-\frac{3\beta}{2}})$. The next step is to show that the term $\Lambda_3(\mu_1, \mu_1, \mu_1)$ is large compared to these.

To do so, we once again decompose the relevant function into two parts. Set

$$\begin{aligned} \mu_3 &= \mu_1 - \mathbb{E}(\mu_1) \\ \text{and } \mu_4 &= \mathbb{E}(\mu_1). \end{aligned}$$

We approximate the expression

$$\Lambda_3(\mu_1, \mu_1, \mu_1) = \Lambda_3(\mu_3 + \mu_4, \mu_3 + \mu_4, \mu_3 + \mu_4)$$

by showing that one term is large compared to the seven others.

It is clear that $\Lambda_3(\mu_4, \mu_4, \mu_4) = \delta^3 N^{3\alpha-3}$. Furthermore, $\widehat{\mu}_3(0) = 0$ and $\widehat{\mu}_3(k) = \widehat{\mu}_1(k)$ for $k > 0$. As in the previous part of the proof, we now use inequality 4.8 to approximate the lesser terms. Firstly,

$$\begin{aligned} |\Lambda_3(\mu_3, \mu_4, \mu_3)| &\leq \|\mu_3\|_{u^2(Z)} \|\mu_4\|_{L^2(Z)} \|\mu_3\|_{L^2(Z)} \\ &= \delta N^{\alpha-1} \left[\max_n \left\{ \left| \left(1 - \frac{n}{K+1}\right) \widehat{\chi}_A(n) \right| \right\} \right] \|\mu_3\|_{L^2(Z)} \\ &= O(\delta N^{\alpha-\frac{\beta}{2}-1} \|\mu_3\|_{L^2(Z)}). \end{aligned}$$

Assuming that $K = O(N^{\frac{1}{3}})$, we can use Parseval and an integral to approximate the $L^2(Z)$ -norm of μ_3 :

$$\begin{aligned} \|\mu_3\|_{L^2(Z)}^2 &\leq \sum_{n=0}^{N-1} |\widehat{\mu}_3(n)|^2 \\ &= O\left(\sum_{n=1}^K \left(1 - \frac{n}{K+1}\right) k^{-\beta} N^{-\beta}\right) \\ &= N^{-\beta} O\left[\int_1^K \left(1 - \frac{x-1}{K+1}\right)^2 (x-1)^{-\beta} dx + \left(1 - \frac{1}{K+1}\right)\right] \\ &= O(N^{-\frac{4\beta}{3}+\frac{1}{3}}). \end{aligned}$$

Therefore, $\|\mu_3\|_{L^2(Z)} = O(N^{-\frac{2\beta}{3}+\frac{1}{6}})$. It follows that the term $|\Lambda_3(\mu_3, \mu_4, \mu_3)|$ is $O(N^{-\frac{19}{6}+\frac{10\alpha}{3}})$ (remembering that $\beta > 2 - 2\alpha$, and the same clearly holds for $|\Lambda_3(\mu_3, \mu_3, \mu_4)|$). Similar calculations show that similar upper bounds hold for every term involving μ_3 . Since $\alpha > 1/2$, this is small compared to $N^{3\alpha-3}$.

All of the approximations now imply that

$$\Lambda_3(\mu, \mu, \mu) = \Omega(N^{3\alpha-3}).$$

The number of arithmetic progressions in Z is counted by the expression

$$N^2 \Lambda_3(\chi_A, \chi_A, \chi_A) - |A|$$

where the second term is employed to ensure we disregard progressions with difference 0. It is important to observe here that the progressions counted is the number of proper progressions (i.e. with non-zero difference) in the cyclic group Z , which may not be equivalent to the number of progressions in the interval $[0, N-1] \subset \mathbb{Z}$ (which will be referred to as *genuine* progressions). The question is now how to eliminate the progressions which “wrap around” the cyclic group Z . In Roth-type theorems, this is often done through density-increment arguments, for instance in chapter 10 of [8]. In our case, we instead consider the set A as a subset of the interval $[0, 3N)$,

which we can again consider as a cyclic group, which we will call Z' . (This is an embedding of A_N into $[0, 3N)$, not a restriction of the original set to a larger interval.) Any proper progression in A , seen as a subset of Z' , would now have to be a genuine progression, since there are no elements of A in the interval $[N, 3N)$. Assuming that there are no progressions except trivial ones, this means that the total number is simply the cardinality of A .

We still denote the characteristic function of A as a subset of Z by χ_A , whereas the characteristic function of A as a subset of Z' is denoted by $\chi_{A'}$. The effect on the Fourier coefficients of χ_A is to “smear” them in such a way that their contribution to the sum-of-squares in the Parseval inequality is taken up by several Fourier coefficients of $\chi_{A'}$. By simply using the definition of the Fourier coefficients, it is easily shown that

$$|\widehat{\chi_{A'}}(3k)|^2 + |\widehat{\chi_{A'}}(3k-1)|^2 + |\widehat{\chi_{A'}}(3k-2)|^2 \leq \frac{1}{3} |\widehat{\chi_A}(k)|^2.$$

This now has the implication that A' satisfies condition (ii) of Theorem 4.1, with some slightly modified constants. It is also obvious that A' has the same fractional density α as A . Thus, the proof implies that the number of three-term arithmetic progressions in A' is greater than

$$cN^{3\alpha-1} - |A'|$$

for some constant c . Since all progressions counted by this expression are genuine, we have established the existence of the required progressions in A .

5 Example of a Salem-type set

In this section we present a version in the whole numbers of the Salem-type set constructed in [3].

Consider the set $\{0, 1, 2, \dots, N^j - 1\}$ for N and j large, and some t , $1 \leq t \leq N$. Our aim is to construct a set which has fractional density $\alpha = \log t / \log N$ (relative to the finite set N^j) and for which the Fourier coefficients of the characteristic function satisfy condition (ii) of Proposition 4.1, with $\beta > 2 - 2\alpha$. At each of the j stages of the construction, we randomly pick a number of points from the total in a ratio t/N , in such a way that the Fourier coefficients of successive sets satisfy certain inequalities.

Let $A_0 = \{0, 1, \dots, N^j - 1\}$. Divide A_0 into N equal intervals (in the whole numbers, as usual) of length N^{j-1} . Let the left-hand endpoints of these intervals be denoted by

$$B_0^* = \{0, N^{j-1}, 2N^{j-1}, \dots, (N-1)N^{j-1}\}.$$

From this set we choose t elements with equal probability $1/t$, and call this B_0 . We form A_1 from this by setting

$$A_1 = \bigcup_{b \in B_0} \{b, b+1, \dots, b+N^{j-1}-1\}.$$

We now divide each interval of A_1 into N equal pieces of length N^{j-2} and form the set

$$B_1^* = \bigcup_{b \in B_0} \{b, b+N^{j-2}, \dots, b+(N-1)N^{j-2}\}$$

from the endpoints of the intervals newly divided. For each of the t components in the union constituting B_1^* , we now have N elements, and from each choose t uniformly and call the resulting

(random) set B_1 . The choice of t elements associated to an element b of B_1^* we call $B_{x(b)}$, whilst the portion of B_1^* of length N^{j-2} starting at b is denoted by $B_{1,b}^*$. Iterating this construction, we obtain from a set A_m consisting of t^m intervals of length N^{j-m} , a subdivision characterised by B_{m+1}^* and a choice of t^{m+1} subintervals characterised by B_{m+1} , which we then use to obtain A_{m+1} .

Some quick calculation will show that this set has fractional density $\log t / \log N$ relative to each interval $[0, N^j]$. In order to determine the rate of decay of the discrete Fourier transform, we borrow the technique utilised in [3], pp. 20–26, adapted to the whole numbers. Fundamental to the calculation is a version of Bernstein’s inequality by Ben Green [1].

Lemma 5.1. *Let X_1, \dots, X_n be independent random variables with $|X_j| \leq 1$, $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_j|^2 = \sigma_j^2$. Let $\sum \sigma_j^2 \leq \sigma^2$, and assume that $\sigma^2 \geq 6n\lambda$. Then*

$$\mathbb{P}\left(\left|\sum_1^n X_j\right| \geq n\lambda\right) \leq 4e^{-n^2\lambda^2/8\sigma^2}.$$

Given a set $B \subset [0, 1]$, we write

$$S_B(k) = \sum_{b \in B} e^{-2\pi i k b}.$$

If we are instead considering a set $B \subset \mathbb{Z}$ with $B \subset [0, N^j]$, we abuse the notation by also using $S_B(k)$ to denote the sum

$$\sum_{b \in B} e^{-\frac{2\pi i k b}{N^j}}.$$

In this way, we can either regard S_B as an exponential sum, or as the Fourier transform of the characteristic function multiplied by a factor N^j .

The previous lemma can be used to prove the following, which is a restatement of Lemma 6.2 in [3]:

Lemma 5.2. *Let $B^* = \{0, \frac{1}{MN}, \frac{2}{MN}, \dots, \frac{N-1}{MN}\}$ and let $1 \leq t \leq N$. Let*

$$\eta^2 t = 32 \log 8N^2 M$$

Then there exists a set $B(x) \subset B^$ with $|B| = t$ such that*

$$\left| \frac{S_{B(x)}(k)}{t} - \frac{S_{B^*}(k)}{N} \right| \leq \eta \text{ for all } k \in [0, MN), x \in [0, N-1],$$

where

$$B(x) = \left\{ \frac{(x+y) \bmod N}{MN} : y \in B \right\}.$$

In the proof of this from Lemma 5.1, it is shown that the condition is satisfied with probability greater than half, indicating that at least half of all possible choices of $B(x)$ will have the property.

One more tool will be necessary before we start the proof – an approximation of the Fourier coefficients by an integral. Specifically, by considering the integral of a smooth function $f : \mathbb{R} \rightarrow \mathbb{C}$ from a to b as being approximated by a left Riemann sum with step-size $\Delta = (b-a)/M$, we get

$$\left| \int_a^b f(x)dx - \Delta \sum_{n=0}^{M-1} f(a+n\Delta) \right| \leq \frac{c(b-a)^3}{M^2} \sup_{x \in [a,b]} |f''(x)|,$$

where the constant c is independent of M , a and b .

We can now use a proof similar to that in [3], with some adjustment for the error term.

Define

$$\psi_m(k) = \frac{N^m}{t^m} \widehat{\chi_{A_m}}(k) = \frac{N^m}{t^m} \left(\frac{1}{N^j} \sum_{a \in A_m} e^{-\frac{2\pi i k a}{N^j}} \right).$$

Although ψ_m is not quite the same as the Fourier transform, it will yield enough information to determine an upper bound.

Let B_m be in relation to A_m as in the construction above. Then

$$\psi_m(k) = \frac{N^m}{t^m} \sum_{b \in B_m} \frac{1}{N^j} \left(e^{-\frac{2\pi i k b}{N^j}} + e^{-\frac{2\pi i k (b+1)}{N^j}} + \dots + e^{-\frac{2\pi i k (b+N^{j-m}-1)}{N^j}} \right) \quad (5.9)$$

Note that if the left-hand endpoint of a subinterval of length N^{j-m-1} is determined, the whole interval is determined. If we consider a choice of t numbers from a collection of N numbers to determine the start of the interval, the exact same choice can be considered to be applied N^{j-m-1} times, from a sample space consisting of translates of the N starting points of the intervals. In the Fourier transform of the characteristic function of the interval, these terms then contribute the same as the starting point, except for a phase shift for each element. If we now wish to compute the difference $|\psi_{m+1} - \psi_m|$, the above expression for ψ_m shows that we can consider the difference

$$\left| \frac{N^{m+1}}{t^m} \sum_{b \in B_m} \left| \frac{S_{B_{m,b}^*}(k)}{N} - \frac{S_{B_{x(b)}}(k)}{t} \right| \left(\frac{1}{N^j} \sum_{n=0}^{N^{j-m-1}-1} e^{-\frac{2\pi i k n}{N^j}} \right) \right| \quad (5.10)$$

In the above, we stay close to the notation of [3] in denoting the exponential sum over the set $B_{m,b}^* = \{b, b + N^{j-m-1}, b + 2N^{j-m-1}, \dots, b + (N-1)N^{j-m-1}\}$ by $S_{B_{m,b}^*}$ and the sum over the corresponding t -choice by $S_{B_{x(b)}}$. We now approximate the final sum by an integral:

$$\frac{1}{N^j} \sum_{n=0}^{N^{j-m-1}-1} e^{-\frac{2\pi i k n}{N^j}} = \int_0^{N^{-(m+1)}} e^{-2\pi i k x} dx + O\left(\frac{k^2 N^{-3(m+1)}}{N^{2j}}\right),$$

where the error term is that of a Riemann sum-approximation of the integral using a step-size N^{-j} .

The error term can easily be shown to be less than the integral in absolute value, especially keeping in mind that we can choose N arbitrarily large. Hence we dispose of it in the absolute value, keeping in mind that it might necessitate the use of a constant $c < 2$, which is not dependent on m . Computing the integral, we find

$$|\psi_{m+1}(k) - \psi_m(k)| \leq c \frac{(1 - e^{-2\pi i k / N^{m+1}})}{t^m (2\pi i k / N^{m+1})} \sum_{b \in B_m} \left| \frac{S_{B_{m,b}^*}(k)}{N} - \frac{S_{B_{x(b)}}(k)}{t} \right|. \quad (5.11)$$

It is now obvious that the above equation is very nearly of the same form as (52) in Lemma 6.4 of [3]. We can therefore apply the result of the lemma to obtain

$$|\psi_{m+1}(k) - \psi_m(k)| \leq 32 \min\left(1, \frac{N^{m+1}}{|k|}\right) t^{-\frac{m+1}{2}} \log(8N^{m+1}). \quad (5.12)$$

We now show that the condition 4.1 (ii) is satisfied for any $\beta > \alpha$ such that $\beta > 2 - 2\alpha$. Since $\psi_0(k) = 0$ for all $k \in \{0, 1, \dots, N^j - 1\}$, we can find an upper bound on $\psi_j(k)$ by bounding the sum of all such differences. By noting that $t = N^\alpha$, we can write the summand as follows (ignoring the constant factor, which has no bearing from here on):

$$\min\left(1, \frac{N^m}{k}\right) t^{-\frac{m}{2}} \log(8N^m) = \min\left(1, \frac{N^m}{k}\right) N^{-\frac{\alpha m}{2}} (\log 8 + m \log N) \quad (5.13)$$

$$= \min\left(1, \frac{N^m}{k}\right) N^{-\frac{\beta m}{2}} N^{-\frac{(\alpha-\beta)m}{2}} (\log 8 + m \log N) \quad (5.14)$$

Using the fact that $N^{-(\alpha-\beta)m/2} j \log N \leq 2(\alpha-\beta)^{-1}$ (which can be established using elementary calculus [3]), the sum is bounded by

$$\sum_{m=1}^j \min\left(1, \frac{N^m}{k}\right) N^{-\frac{\beta m}{2}} \left(N^{-\frac{(\alpha-\beta)m}{2}} \log 8 + 2(\alpha-\beta)^{-1}\right) \leq \sum_{m=1}^j \min\left(1, \frac{N^m}{k}\right) N^{-\frac{\beta m}{2}} (\log 8 + 2(\alpha-\beta)^{-1}) \quad (5.15)$$

We consider two different regions: one where $1 \leq m \leq \log k / \log N$ and one where $m > \log k / \log N$. In the first case,

$$S_1 = k^{-1} t^j (\log 8 + 2(\alpha-\beta)^{-1}) \sum_{1 \leq m \leq \frac{\log k}{\log N}} N^{m(1-\frac{\beta}{2})} \quad (5.16)$$

The sum on the right is easily bounded, thus

$$S_1 \leq 2k^{-1} (\log 8 + 2(\alpha-\beta)^{-1}) k^{1-\frac{\beta}{2}} \leq C_1 k^{-\beta/2} \quad (5.17)$$

for some C_1 independent of N, j .

Approximating the second part of the sum is similar, and we obtain

$$S_2 = (\log 8 + 2(\alpha-\beta)^{-1}) \sum_{\frac{\log k}{\log N} < m \leq j} N^{-\beta m/2} \leq C_2 k^{-\beta/2}. \quad (5.18)$$

Using the bounds for S_1 and S_2 , we get

$$|\psi_j(k)| \leq C |k|^{-\beta/2}.$$

We can obtain $\widehat{\chi_{A_j}}(k)$ by multiplication of $\psi_j(k)$ by a factor t^j / N^j . Because of the construction,

$$t^j / N^j = N^{(\alpha-1)j} < N^{-\frac{\beta j}{2}},$$

since we chose $\beta > 2 - 2\alpha$. This yields the desired bound on the Fourier coefficients.

By this example and the result in the previous section, there seems to be a clear correspondence between perfect subsets of $[0, 1]$ and sets in \mathbb{Z} , which preserves Hausdorff- and Fourier-dimensional properties. An examination of the precision of the correspondence will appear in the sequel to this paper.

References

- [1] B. Green, *Arithmetic progressions in sumsets*, Geom. Funct. Anal. **12** (2002), 584-597
- [2] B. Green, T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. Math. **167** (2008), 481-547
- [3] I. Laba, M. Pramanik, *Arithmetic progressions in sets of fractional dimension*, Geom. Funct. Anal. **19** (2009), 2, 429-456
- [4] S.C. Leth, *Applications of nonstandard models and Lebesgue measure to sequences of natural numbers*, Trans. Amer. Math. Soc. **302** (1988), 2, 457-468
- [5] N. Lyall, *Roth's theorem - The Fourier analytic approach*, available at <http://www.math.uga.edu/~lyall/REU/Roth.pdf>
- [6] P. Potgieter, *Nonstandard analysis, fractal properties and Brownian motion*, Fractals **17** (2009), 117-129
- [7] K. Roth, *On certain sets of integers*, J. London Math. Soc. **28** (1953), 245-252
- [8] T. Tao, V. Vu, *Additive combinatorics*, Cambridge University Press, 2006